FLOW OF A GAS STREAM FROM A CONTAINER OF FINITE WIDTH AT MAXIMUM DISCHARGE

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We deal with a case of plane laminar gas flow. It is an adiabatic steady-state flow without vorticity. The method for solving such flows in the sub-sonic case with a specified reference velocity, has been given by Chaplygin [1]. Fal'kovich extended the Chaplygin method to flows involving more than one characteristic velocity [2]. The solution described here is based on this extension and also makes use of Frankel's results [3], where it is demonstrated that the flow problem of a gaseous stream at maximum discharge reduces to the Tricomi problem for the Chaplygin equation [4]

$$4\tau^2(1-\tau)\frac{\partial^2\psi}{\partial\tau^2}+4\tau[1+(\beta-1)\tau]\frac{\partial\psi}{\partial\tau}+[1-(2\beta+1)\tau]\frac{\partial^2\psi}{\partial\theta^2}=0 \qquad (0.1)$$

1. We assume a flat container of finite width $2B$ with straight parallel walls (see Fig. 1) and an opening of width $2b$. We envisage a maximum discharge gas flow from the vessel $\rho_0 Q$. The flow has an axis of symmetry and is mixed sub and super-sonic. C_1C_2 is the sonic line, and D is the center of the nozzle. The flow region to the left of the characteristics DC_1 and DC_2 corresponds in the hodograph plane in Fig. 2 to epicycloids C_1C_1', DC_1', DC_2', C_2C_2'. The coordinates in the hodograph plane are the polars $\tau = v^2/v_{\max}^2$ and angle θ of inclination of velocity to the x-axis. The region has a straight section of magnitude τ_0 along the horizontal axis.

We will assume that along the streamline AB_2C_2E the stream function is $\psi = +1/2\ Q$, then along the streamline AB_1C_1E it has the value $-Q/2$. In the hodograph plane we have the following

$$\psi=-\tfrac{1}{2}Q, \quad \tau_0>\tau\geqslant 0, \qquad \theta=+0 \qquad (1.1)$$

$$\psi=-\tfrac{1}{2}Q, \quad 0\leqslant\tau\leqslant\tau_*, \qquad \theta=\tfrac{1}{2}\pi \qquad (1.2)$$

$$\psi = -\tfrac{1}{2}Q,\ \theta + f(\lambda) = \tfrac{1}{2}\pi, \quad \tfrac{1}{2}\pi \geqslant \theta \geqslant \tfrac{1}{4}\pi, \tag{1.3}$$

$$\psi = 0, \qquad \tau_0 < \tau < \tau_*, \qquad \theta = 0 \tag{1.4}$$

$$\psi = \tfrac{1}{2}Q, \qquad \tau_0 > \tau \geqslant 0, \qquad \theta = -0$$

$$\psi = \tfrac{1}{2}Q, \qquad 0 \leqslant \tau \leqslant \tau_*, \qquad \theta = -\tfrac{1}{2}\pi$$

$$\psi = \tfrac{1}{2}Q, \qquad \theta - f(\lambda) = -\tfrac{1}{2}\pi, \qquad -\tfrac{1}{2}\pi \leqslant \theta \leqslant -\tfrac{1}{4}\pi$$

$$f(\lambda) = h \operatorname{arc\,tg} \sqrt{\frac{\lambda^2 - 1}{h^2 - \lambda^2}} - \operatorname{arc\,tg} h \sqrt{\frac{\lambda^2 - 1}{h^2 - \lambda^2}},$$

$$\lambda = \sqrt{\frac{\varkappa + 1}{\varkappa - 1}}\,\tau, \qquad h^2 = \frac{\varkappa + 1}{\varkappa - 1}$$

Because of symmetry it is sufficient to deal with the upper half of the hodograph plane. We arrive at the following Tricomi problem for solving the flow; it is required to find a solution for equation (0.1) within the region bounded by straight lines B_1C_1 and B_1D and the characteristics (C_1C_1' and DC_1') which take the given values (1.1) to (1.4) at the boundaries of the region.

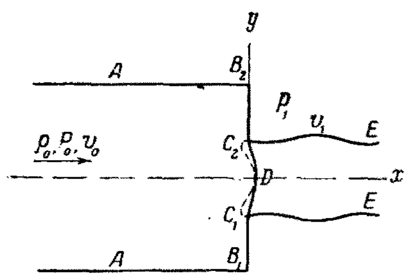

Fig. 1.

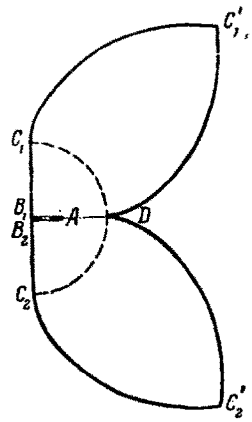

Fig. 2.

If $\psi(\tau, \theta)$ has been found we transform to the flow plane by means of the formulas:

$$x = \frac{(1-\tau)^{-\beta}}{v} \int_0^\theta \left(2\tau \frac{\partial\psi}{\partial\tau} \cos\theta - \frac{\partial\psi}{\partial\theta} \sin\theta\right) d\theta + x_0(\tau)$$

$$y = \frac{(1-\tau)^{-\beta}}{v} \int_0^\theta \left(2\tau \frac{\partial\psi}{\partial\tau} \sin\theta + \frac{\partial\psi}{\partial\theta} \cos\theta\right) d\theta + y_0(\tau) \tag{1.5}$$

2. Having drawn the circular arc AH, radius τ_0 with center at the origin, we divide the region where the solution is determined into two parts.

We have the solution in region *1* (Fig. 3) as follows

$$\psi_1(\tau, \theta) = -\frac{Q}{2} + \sum_{n=1}^{\infty} \alpha_n z_n(\tau) \sin 2n\theta \tag{2.1}$$

where $z_n(\tau)$ is the Chaplygin function [1].

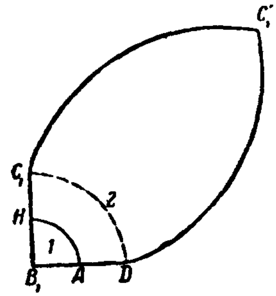

Fig. 3.

The solution in region 2 is as follows

$$\psi_2(\tau, \theta) = -\frac{Q}{\pi}\theta + \sum_{n=1}^{\infty} [A_n z_n(\tau) + B_n \zeta_n(\tau)] \sin 2n\theta \tag{2.2}$$

where $\zeta_n(\tau) = \psi^*_{-2n}(\tau)$ is the Cherry function [2,5]; the function $\psi_1(\tau, \theta)$ satisfies conditions (1.1) and (1.2), the function $\psi_2(\tau, \theta)$ satisfies (1.2) and (1.4), whilst $\psi_2(\tau, \theta)$ should be an analytic continuation of function $\psi_1(\tau, \theta)$ in region 2. Therefore on arc *AH*

$$\psi_1(\tau_0, \theta) = \psi_2(\tau_0, \theta), \qquad \left(\frac{\partial\psi_1}{\partial\tau}\right)_{\tau=\tau_0} = \left(\frac{\partial\psi_2}{\partial\tau}\right)_{\tau=\tau_0} \tag{2.3}$$

These conditions yield a system of equations for finding the unknown coefficients, from which we can obtain

$$B_n = -\frac{Q}{n\pi}\frac{z_n'(\tau_0)}{W(\tau_0)}, \qquad A_n - \alpha_n = \frac{Q}{n\pi}\frac{\zeta_n'(\tau_0)}{W(\tau_0)}, \qquad W(\tau_0) = n\frac{(1-\tau_0)^\beta_*}{\tau_0} \tag{2.4}$$

where $W(\tau_0)$ is the value of the Vronskian at $\tau = \tau_0$, built up from functions $z_n(\tau)$ and $\zeta_n(\tau)$. On inserting, we get

$$B_n = -\frac{Q\tau_0}{\pi(1-\tau_0)^\beta}\frac{z_n'(\tau_0)}{n^2}, \qquad \alpha_n = A_n - \frac{Q\tau_0}{\pi(1-\tau_0)^\beta}\frac{\zeta_n'(\tau_0)}{n^2} \tag{2.5}$$

Expression (2.2) can now be written

$$\psi_2(\tau, \theta) = -\frac{Q}{\pi}\theta + \sum_{n=1}^{\infty} A_n z_n(\tau) \sin 2n\theta - \frac{Q}{\pi}\frac{\tau_0}{(1-\tau_0)^\beta}\sum_{n=1}^{\infty} z_n'(\tau_0)\zeta_n(\tau)\frac{\sin 2n\theta}{n^2} \tag{2.6}$$

We now take into consideration the condition which the stream function satisfies in the neighborhood of the center of the nozzle, and represent coefficients A_n in this form

$$A_n = \frac{Q}{\pi}\,\frac{a_0}{n^{4/3} z_n(\tau_*)} + \frac{Q}{\pi}\,\frac{a_n}{z_n(\tau_*)} \tag{2.7}$$

Frankel' gave a similar representation of flow from an infinitely wide container [7]. Taking (2.7) into account, expression (2.6) takes the following form

$$\frac{\pi}{Q}\,\psi_2 = -\theta + a_0\psi_2^\circ + \delta\psi_2 \tag{2.8}$$

where

$$\psi_2^\circ = \sum_{n=1}^{\infty} \frac{z_n(\tau)}{z_n(\tau_*)}\,\frac{\sin 2n\,\theta}{n^{4/3}} \tag{2.9}$$

$$\delta\psi_2 = \sum_{n=1}^{\infty} a_n \frac{z_n(\tau)}{z_n(\tau_*)} \sin 2n\,\theta - \frac{\tau_0}{(1-\tau_0)^{\beta}} \sum_{n=1}^{\infty} z_n'(\tau_0)\,\zeta_n(\tau)\,\frac{\sin 2n\theta}{n^{2}} \tag{2.10}$$

It is not possible to determine exactly all the unknown coefficients from condition (1.3) on characteristic C_1C_1' To find several of the coefficients a_n approximately we have chosen a method in which the boundary conditions (1.3) are satisfied at several points on the characteristic C_1C_1'

If coefficients a_n are determined, equations (2.1) and (2.8) yield a solution of the required boundary value problem. Having solved the Tricomi problem we can find the relation between the mass flow of the gas Q and the containers parameter b/B. Let us use (1.5); we will integrate it along the sonic line (circular arc DC_2). On the sonic line $\tau = \tau_*$ when $\theta = 0$, $y = 0$. Therefore $y_0(\tau) = 0$. When $\theta = -\pi/2$, $y = b$. Thus

$$b = \frac{(1-\tau_*)^{-\beta}}{a_*} \int_0^{-1/2\pi} \left(2\tau_* \frac{\partial\psi_2}{\partial\tau}\bigg|_{\tau=\tau_*} \sin\theta + \frac{\partial\psi_2}{\partial\theta}\bigg|_{\tau=\tau_*} \cos\theta\right) d\theta$$

$$b = \frac{Q(1-\tau_*)^{-\beta}}{\pi a_*} \int_0^{-1/2\pi} [f_1(\theta)\cos\theta + 2\tau_* f_2(\theta)\sin\theta]\,d\theta \tag{2.11}$$

where

$$f_1(\theta) = \frac{\pi}{Q}\,\frac{\partial\psi_2}{\partial\theta}\bigg|_{\tau=\tau^*}, \qquad f_2(\theta) = \frac{\pi}{Q}\,\frac{\partial\psi_2}{\partial\tau}\bigg|_{\tau=\tau_*}$$

We now find functions $f_1(\theta)$ and $f_2(\theta)$:

$$f_1(\theta) = -1 + a_0 \frac{\partial\psi_2^\circ}{\partial\theta}\bigg|_{\tau=\tau_*} + 2\sum_{n=1}^{\infty} n a_n \cos 2n\theta - \frac{2\tau_0}{(1-\tau_0)^{\beta}} \sum_{n=1}^{\infty} z_n'(\tau_0)\,\zeta_n(\tau_*)\,\frac{\cos 2n\theta}{n} \tag{2.12}$$

$$f_2(\theta)=a_0\left.\frac{\partial\psi_2^\circ}{\partial\tau}\right|_{\tau=\tau_*}+\sum_{n=1}^{\infty}a_n\frac{z_n'(\tau_*)}{z_n(\tau_*)}\sin 2n\theta-\frac{\tau_0}{(1-\tau_0)^{\beta}}\sum_{n=1}^{\infty}z_n'(\tau_0)\,\zeta_n'(\tau_*)\frac{\sin 2n\theta}{n^2} \tag{2.13}$$

$$\left.\frac{\partial\psi_2^\circ}{\partial\theta}\right|_{\tau=\tau_*}=2\sum_{n=1}^{\infty}\frac{\cos 2n\theta}{n^{1/3}},\qquad \left.\frac{\partial\psi_2^\circ}{\partial\tau}\right|_{\tau=\tau^*}=\frac{1}{\tau_*}\sum_{n=1}^{\infty}X_n^*\,\frac{\sin 2n\theta}{n^{1/3}},\qquad X_n^*=\frac{\tau_*\,z_n'(\tau_*)}{n\,z_n(\tau_*)} \tag{2.14}$$

where $X_n(\tau_*)$ is the value of the auxiliary Chaplygin function [1]. Frankel' obtained an asymptotic formula for X_n^* [6, 7]

$$X_n^*=-\frac{1}{2}\left(\frac{2}{\varkappa+1}\right)^{\frac{1}{\varkappa-1}}(C_0n^{-1/3}+C_1n^{-1}+C_2n^{-5/3}+C_3n^{-7/3})+\delta X_n^*,\quad \delta X_n^*=0\,(n^{-3}) \tag{2.15}$$

Substituting (2.15) into (2.14), we obtain

$$\left.\frac{\partial\psi_2^\circ}{\partial\tau}\right|_{\tau=\tau^*}=-\frac{1}{2\tau_*}\left(\frac{2}{\varkappa+1}\right)^{\frac{1}{\varkappa-1}}\left(C_0\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^{2/3}}+C_1\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^{4/3}}+C_2\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^2}+\right.$$
$$\left.+C_3\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^{8/3}}\right)+\frac{1}{\tau_*}\sum_{n=1}^{\infty}\delta X_n^*\,\frac{\sin 2n\theta}{n^{1/3}} \tag{2.16}$$

We introduce the following integrals

$$I_1=\int_0^{-1/2\pi}f_1(\theta)\cos(\theta)\,d\theta,\qquad I_2=\int_0^{-1/2\pi}f_2(\theta)\sin\theta\,d\theta$$

The calculated values of which are

$$I_1=1+2a_0\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}(4n^2-1)}+2\sum_{n=1}^{\infty}\frac{(-1)^n\,na_n}{4n^2-1}-$$
$$-\frac{2\tau_0}{(1-\tau_0)^{\beta}}\sum_{n=1}^{\infty}z_n'(\tau_0)\,\zeta_n(\tau_*)\,\frac{(-1)^n}{n\,(4n^2-1)} \tag{2.17}$$

$$I_2=-\frac{a_0}{\tau_*}\left(\frac{2}{\varkappa+1}\right)^{\frac{1}{\varkappa-1}}\left(C_0\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{-1/3}(4n^2-1)}+C_1\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}(4n^2-1)}+\right.$$
$$\left.+C_2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,(4n^2-1)}+C_3\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{5/3}(4n^2-1)}\right)+\frac{2a_0}{\tau_*}\sum_{n=1}^{\infty}\frac{(-1)^n\,\delta X_n^*\,n^{2/3}}{4n^2-1}+$$
$$+2\sum_{n=1}^{\infty}a_n\frac{z_n'(\tau_*)\,n\,(-1)^n}{z_n(\tau_*)\,(4n^2-1)}-\frac{2\tau_0}{(1-\tau_0)^{\beta}}\sum_{n=1}^{\infty}z_n'(\tau_0)\,\zeta_n'(\tau_*)\,\frac{(-1)^n}{n\,(4n^2-1)} \tag{2.18}$$

Expression (2.11) becomes as follows:

$$b = \frac{Q(1-\tau_*)^{-\beta}}{\pi a_*}\,(I_1 + 2\tau_* I_2] \tag{2.19}$$

Owing to the fact that $2Bv_0 = Q$, we get from (2.19)

$$\frac{b}{B} = 2\sqrt{\frac{\varkappa+1}{\varkappa-1}}\,\tau_0\,(I_1 + 2\tau_* I_2)\,\frac{1}{\pi(1-\tau_*)^{\beta}} \tag{2.20}$$

Let us find the coefficient of discharge:

$$\varepsilon = \frac{\rho_0 Q}{2b\rho_* a_*} = \frac{\pi(1-\tau_0)^{\beta}}{2(I_1 + 2\tau_* I_2)} \tag{2.21}$$

If we compute several values of τ_0 from formulas (2.20) and (2.21) we can construct the relation $\epsilon = \epsilon(b/B)$. Bear in mind that the limiting ordinates are known: $\epsilon(0) = 0.85$ (cm) [7] and $\epsilon(1) = 1$.

3. In order to determine the coefficients a_n it is essential to be able to work out $\psi_2{}^0$ on the characteristic $C_1C_1{}'$. The series (2.9) is not suitable for this purpose because it only converges slowly for $\tau > \tau_*$. To find $\psi_2{}^0$ we use the asymptotic formula, obtained by Aslanov [8], for the relation $z_n(\tau)/z_n(\tau_*)$ for $\tau \geqslant \tau_*$.

From this it follows that

$$\xi_n(\tau) = \frac{z_n(\tau)}{z_n(\tau_*)} = \xi_n^{\circ}(\tau) + \delta\xi_n \tag{3.1}$$

where

$$\xi_n{}^{\circ}(\tau) = A\,|V(\tau)|\,n^{-1/6}\sin\left[\frac{4n}{3}(-\eta)^{3/2} + \frac{\pi}{4}\right] \tag{3.2}$$

$$A = \left(\frac{\varkappa+1}{2}\right)^{1/6}\frac{3^{2/3}\,\Gamma(2/3)}{(1-\tau_*)^{\beta/2}\sqrt{\pi}} = 2.05198$$

$$V(\tau) = \left[\frac{(1-\tau)^{2\beta+1}}{1-\tau/\tau_*}\right]^{1/4}, \quad \eta = \left[\frac{3}{4}\int_{\tau}^{\tau_*}\sqrt{\frac{1-\tau/\tau_*}{1-\tau}}\,\frac{d\tau}{\tau}\right]^{2/3}$$

If we insert (3.1) and (3.2) into (2.9) we arrive at the following expression for $\psi_2{}^0(\tau)$ for $\tau \geqslant \tau_*$.

$$\psi_2{}^{\circ} = A\,|V(\tau)|\sum_{n=1}^{\infty}\frac{\sin 2n\theta\,\sin[4/3\,n\,(-\eta)^{3/2} + 1/4\,\pi]}{n^{7/6}} + \sum_{n=1}^{\infty}\delta\xi_n\frac{\sin 2n\theta}{n^{4/3}}$$

which can be put in the following form

$$\psi_2{}^{\circ} = \frac{A\,|V(\tau)|}{2\sqrt{2}}\left\{\sum_{n=1}^{\infty}\frac{\sin[2\theta + 4/3\,(-\eta)^{3/2}]\,n}{n^{7/6}} + \sum_{n=1}^{\infty}\frac{\sin[2\theta - 4/3\,(-\eta)^{3/2}]\,n}{n^{7/6}} - \right.$$

$$\left. - \sum_{n=1}^{\infty}\frac{\cos[2\theta + 4/3\,(-\eta)^{3/2}]\,n}{n^{7/6}} + \sum_{n=1}^{\infty}\frac{\cos[2\theta - 4/3\,(-\eta)^{3/2}]\,n}{n^{7/6}}\right\} + \sum_{n=1}^{\infty}\delta\xi_n\frac{\sin 2n\theta}{n^{4/3}} \tag{3.3}$$

Calculating $(-\eta)^{3/2}$ we have

$$(-\eta)^{3/2} = \frac{3}{2}\left(\frac{1}{\sqrt{\tau_*}} \operatorname{arc\,tg} \sqrt{\frac{\tau - \tau_*}{1-\tau}} - \operatorname{arc\,tg} \sqrt{\frac{\tau/\tau_* - 1}{1-\tau}}\right) = \frac{3}{2} f(\lambda) \qquad (3.4)$$

From (3.4) and $\theta + f(\lambda) = \pi/2$, the equation of the characteristic C_1C_1', it follows that on this characteristic

$$2\theta + \frac{4}{3}(-\eta)^{3/2} = \pi, \qquad 2\theta - \frac{4}{3}(-\eta)^{3/2} = 4\theta - \pi$$

Therefore, expression (3.3), for points on the characteristic C_1C_1', can be written thus:

$$\psi_2^\circ = \frac{A\,|V(\tau)|}{2\sqrt{2}}\left[\sum_{n=1}^{\infty} \frac{\sin(4\theta-\pi)\,n}{n^{5/2}} + \sum_{n=1}^{\infty} \frac{\cos(4\theta-\pi)\,n}{n^{5/2}} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{5/2}}\right] + \sum_{n=1}^{\infty} \delta\xi_n \frac{\sin 2n\theta}{n^{4/3}} \qquad (3.5)$$

4. The trigonometric series (3.5) must be made more rapidly convergent. We will improve the convergency by making use of Lindelof's formula [9]

$$\sum_{n=1}^{\infty} \frac{x^n}{n^s} = \Gamma(1-s)\,(-\ln x)^{s-1} + \sum_{n=0}^{\infty} \zeta(s-n)\frac{(\ln x)^n}{n!} \qquad (4.1)$$

which is valid in the complex x-plane with a cut on the real axis from 1 to ∞, for $\operatorname{Re}(S) > 1$ and S not an integer.

From formula (4.1), for $|x| = 1$ and $s - 1 = p/q$, we can get

$$\sum_{n=1}^{\infty} \frac{\cos n\varphi}{n^s} = \Gamma(1-s)\,\varphi^{s-1} \cos\frac{3p\pi + 4\pi k}{2q} + \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(s-2n)}{(2n)!}\varphi^{2n} \qquad (4.2)$$

$$\sum_{n=1}^{\infty} \frac{\sin n\varphi}{n^s} = \Gamma(1-s)\,\varphi^{s-1} \sin\frac{3p\pi + 4\pi k}{2q} + \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(s-2n-1)}{(2n+1)!}\varphi^{2n+1} \qquad (4.3)$$

In formulas (4.2) and (4.3) k takes one of the values 0, 1, 2, ..., $q - 1$.

We can get an expression for the main term in expansion (4.3) in a different way with different considerations, as was done, for instance by Usubakunov*. A comparison gives the following:

$$\Gamma(1-s)\sin\frac{3p\pi + 4\pi k}{2q} = \frac{1}{\Gamma(s)}\,\frac{\pi}{2\sin{}^1/_2\,\pi s}$$

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* Plane-parallel flow at velocities above critical through a slit from a channel of finite width (Contribution to the theory of labyrinth packings in steam turbines and slit type weirs) Dissertation for candidate's degree Physics and Mathematics, Frunze 1955.

It is then possible to find k, which is important if it is necessary to use formula (4.2) for computations. We get the following formulas for the series which we are after:

$$\sum_{n=1}^{\infty} \frac{\cos n\varphi}{n^{3/2}} = -\sqrt{2\pi}\,\varphi^{1/2} + \sum_{n=0}^{\infty} (-1)^n \frac{\zeta\,(3/2 - 2n)}{(2n)!}\,\varphi^{2n} \qquad (4.4)$$

$$\sum_{n=1}^{\infty} \frac{\sin n\varphi}{n^{3/2}} = \sqrt{2\pi}\,\varphi^{1/2} + \sum_{n=0}^{\infty} (-1)^n \frac{\zeta\,(3/2 - 2n - 1)}{(2n + 1)!}\,\varphi^{2n+1} \qquad (4.5)$$

The series on the right-hand side of the formulas are absolutely convergent for $|\phi| < 2\pi$.

5. Now let us put $\tau_0 = 0.04$ ($M_0 = 0.456$, $p_1/p_0 = 0.0447$). We will determine the coefficients a_n by satisfying the boundary condition (1.3) at five points on the characteristic C_1C_1'. We choose the following points on the characteristic

$$\begin{array}{llllll} \tau_j = & +0.26 & 0.32 & 0.38 & 0.44 & 0.50 \\ \theta_j = & 83°\,7.70 & 77°5.63' & 70°42.58' & 64°6.91' & 57°20.27' \end{array}$$

To find $a_n (n = 0, 1, 2, 3, 4)$ we have this system of equations

$$a_0\psi_2^\circ(\tau_j, \theta_j) + a_1 \frac{z_1(\tau_j)}{z_1(\tau_*)} \sin 2\theta_j + \ldots + a_4 \frac{z_4(\tau_j)}{z_4(\tau_*)} \sin 8\theta_j = \theta_j - \frac{\pi}{2} +$$
$$+ \frac{\tau_0}{(1-\tau_0)^\beta} \sum_{n-1}^{\infty} z_n'(\tau_0)\zeta_n(\tau_j) \frac{\sin 2n\,\theta_j}{n^2} \quad (j=1,2,3,4,5) \qquad (5.1)$$

The coefficients on the left-hand side of (5.1) are worked out from tables of functions and formulas (3.5), (4.4) and (4.5).

Determination of the elements of the last column of the expanded matrix in system (5.1) requires the ability to work out the values of Cherry functions $\zeta_n(\tau)$. Knowledge of the Vronskian functions $z_n(\tau)$, $\zeta_n(\tau)$ gives the differential equation

$$\begin{vmatrix} z_n'(\tau) & \zeta_n'(\tau) \\ z_n(\tau) & \zeta_n(\tau) \end{vmatrix} = n \frac{(1-\tau)^\beta}{\tau}$$

from which $\zeta_n(\tau)$ can be evaluated up to an arbitrary constant. As function $\zeta_n(\tau)$ we will take the following

$$\zeta_n(\tau) = -n z_n(\tau) \int_{\tau^*}^{\tau} \frac{(1-t)^\beta}{t z_n^2(t)}\,dt \qquad (5.2)$$

Calculations with Simpson's rule give the following values for $\zeta_n(\tau)$ and $\zeta_n'(\tau_*)$:

TABLE 1.

τ	ζ_1	ζ_2	ζ_3	ζ_4
1	0	0	0	0
0.26	—2.30417	—43.9200	—625.110	—7796.64
0.32	—3.04486	—53.3260	—650.808	—6346.84
0.38	—3.41703	—51.3526	—451.938	14813.72
0.44	—3.52153	—41.4922	754.233	71067.36
0.50	—3.43409	—27.6224	5011.317	60575.12
	ζ_1'	ζ_2'	ζ_3'	ζ_4'
$\frac{1}{6}$	—35.6002	—698.478	—10468.7	—140867

The expanded matrix of the system of equations (5.1):

TABLE 2.

0.15832	0.28126	—0.58862	0.85217	—1.01807	—0.130214
0.31305	0.52489	—0.93088	0.95512	—0.61015	—0.253112
0.46084	0.73162	—0.92329	0.39706	0.06136	—0.389836
0.62010	0.86567	—0.61092	—0.03306	—0.25588	—0.506616
0.78187	0.91362	—0.23207	+0.10838	—0.46293	—0.664797

The following values of a_n are obtained:

$$a_0 = 3.22389, \quad a_1 = -3.80398, \quad a_2 = -1.20411, \quad a_3 = -0.49295, \quad a_4 = -0.13810$$

From formulas (2.17) and (2.18) the following values are calculated

$$I_1 = 1.38530, \qquad I_2 = 0.41096$$

From formulas (2.20) and (2.21) we find the vessel parameter ratio and coefficient of discharge:

$$b / B = 0.749, \ \varepsilon = 0.932$$

This flow problem was solved approximately by Usubakunov in 1955 using the Tricomi equation as a basis. He found that $\epsilon = 0.936$ when $b/B = 0.64$. This contradicts the result derived above, for it is obvious that function $\epsilon = \epsilon\,(b/B)$ should be monotonically increasing.

Results more accurate than those obtained here can be achieved by greater accuracy in satisfying the boundary condition on the characteristic and greater precision in evaluating $\zeta_n(\tau)$.

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